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Ed:

Given recent interest in my PhD Thesis (Shipley, 1978) and your posting of it to the internet, I am providing you with errata and notes from my copy of the document. After the fog cleared, it looks like I went through the derivation in June 1979 and found typos and a few improvements to the formulae. Some of what follows is an expansion of equations where we normally state "it follows that." Please feel free to post this with the thesis hyperlink.

Regards,

Scott T. Shipley

identity This is opposite page 77 (Eq. 3.22)

$$\int_{-\infty}^{+\infty} e^{-ax^2 \pm bx}\,dx = \int_0^{+\infty} e^{-ax^2 \pm bx}\,dx + \int_0^{+\infty} e^{-ax^2 \mp bx}\,dx$$

see A&S 7.4.32

$$= \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right)\cdot\left\{\operatorname{erf}\left(\sqrt{a}\,x \pm \frac{b}{2\sqrt{a}}\right)\Big|_0^{\infty} + \operatorname{erf}\left(\sqrt{a}\,x \mp \frac{b}{2\sqrt{a}}\right)\Big|_0^{\infty}\right\}$$

$$\lim_{z\to\infty} \operatorname{erf}(z) = 1$$

$$\operatorname{erf}(-z) = -\operatorname{erf}(z)$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right)\cdot\left[1 - \operatorname{erf}\left(\pm\frac{b}{2\sqrt{a}}\right) + 1 - \operatorname{erf}\left(\mp\frac{b}{2\sqrt{a}}\right)\right]$$

$$= \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right)$$

solution to Eq. 3.22

general linear equation of first order:

$$y' + y\,a(x) = b(x)$$

then

$$y(x) = e^{-A(x)}\left(B(x) + c\right)$$

where

$$A(x) = \int_{x_0}^{x} a(t)\,dt$$

$$B(x) = \int_{x_0}^{x} e^{A(t)}\,b(t)\,dt$$

$$\therefore \quad \hat{\imath}(x) = e^{\int_{x_1}^{x}[-\beta + \hat{j}(\eta p + \xi q)]dx} \int_{x_0}^{x} e^{-\int_{x_1}^{x'}[\beta - \hat{j}(\eta p + \xi q)]dx''}\left(b(x') + c\right)dx'$$

for region before cloud boundary, $b(x) = 0$ and $c = 1$

after cloud boundary, $c = 0$.

initial value, let

$$\hat{I}_0 = \frac{\gamma_t}{\pi} \exp[-\gamma_t(\eta^2+\zeta^2)] \cdot \delta(\eta - y/x)\, \delta(\zeta - z/x)$$

$$\rightarrow \iint_{-\infty}^{+\infty} \hat{I}_0 \, d\eta \, d\zeta = \delta(\eta - y/x)\, \delta(\zeta - z/x) \longrightarrow \text{normalized to } \underline{\text{one}}$$

then

$$\hat{i}_0 = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \hat{I}_0 \exp(\hat{j}(py + qz))\, dy\, dz$$

$$= \frac{\gamma_t}{2\pi^2} \exp[-\gamma_t(\eta^2+\zeta^2)] \cdot \iint_{-\infty}^{+\infty} \exp(\hat{j}py + \hat{j}qz)\, \delta(\eta - y/x)\, \delta(\zeta - z/x)\, dy\, dz$$

$$= \frac{\gamma_t}{2\pi^2} \exp[-\gamma_t(\eta^2+\zeta^2)] \; \exp[+\hat{j}p\eta x \;+ \hat{j}q\zeta x]$$

∴ Eq. 3.23

This is opposite page 78 (Eq. 3.23)

22 June 79, let $\gamma_{b,s} = \epsilon$, $\gamma_{f,i_2} = \gamma_2$, $\gamma_{f,i_1} = \gamma_1$, $\gamma_t = \eta$

$$\frac{1}{R^2\gamma_2\gamma_1\eta}\Delta_{2,s,0}B_{2,s,0} = \left(\frac{R-R_c}{R}\right)^2\left\{\frac{1}{\gamma_1}u_1^2 + \frac{1}{\gamma_2}u_2^2 + \frac{\epsilon}{\gamma_1\gamma_2}(u_1-u_2)^2\right\}$$
$$+\frac{1}{\eta}\left\{1 + \frac{\epsilon}{\gamma_1}\left(1-\frac{R-R_c}{R}u_1\right)^2 + \frac{\epsilon}{\gamma_2}\left(1-\frac{R-R_c}{R}u_2\right)^2\right\}$$

this checks 3.37a & 3.37b for $n=2, m=0$.

$$\alpha_N = \frac{\dfrac{\prod_{i=1}^{N-1} a(u_i)p(u_i)}{C_N + \langle\Theta_t^2\rangle D_N}\displaystyle\int_0^{\psi}\theta d\theta \exp\left\{-\frac{D_N\theta^2 + E_N\Theta_L^2}{C_N+\langle\Theta_t^2\rangle D_N}\right\}\cdot I_0\left\{\frac{2\Theta_L\theta D_N^*}{C_N+\langle\Theta_t^2\rangle D_N}\right\}}{\dfrac{1}{\langle\Theta_t^2\rangle}\exp\left\{-\left(\dfrac{1}{\langle\Theta_t^2\rangle}+\dfrac{1}{\langle\Theta_b^2\rangle}\right)\Theta_L^2\right\}\displaystyle\int_0^{\psi}\theta d\theta \exp\left\{-\frac{\theta^2}{\langle\Theta_t^2\rangle}\right\}\cdot I_0\left\{\frac{2\Theta_L\theta}{\langle\Theta_t^2\rangle}\right\}}$$

where $\Theta_L = \dfrac{L}{R}$ (L is transmitter-receiver lateral separation)

$$D_N^* = 1 + \sum_{i=1}^{N-1}\frac{\langle\Theta_f^2\rangle_{u_i}}{\langle\Theta_b^2\rangle}\left(1-\frac{R-R_c}{R}u_i\right)$$

This is opposite page 82 (Eq. 3.36)

$$\frac{P_{n,m}}{P_1} = \tau^{n+m}\int_0^1 du_1\int_0^{u_1} du_2\cdots\int_0^{u_{n-1}} du_n\cdot\int_0^1 du_{n+1}\int_{u_{n+1}}^1 du_{n+2}\cdots\int_{u_{n+m-1}}^1 du_{n+m}\cdot\mathcal{I}_{n+m+1} \qquad (3.35)$$

where

$$\mathcal{I}_n = \frac{\prod_{i=1}^{N} a(u_i)\rho(u_i)}{C_N+\langle\Theta_t^2\rangle D_N}\int_0^{\psi}\theta d\theta\, \exp\left\{\frac{-D_N}{C_N+\langle\Theta_t^2\rangle D_N}\left[\theta^2 + \frac{L^2}{D_N E_N}\right]\right\}\cdot$$

$$\cdot I_o\left\{\frac{2L\theta}{C_N+\langle\Theta_t^2\rangle D_N}\left[\frac{D_N}{E_N} - \frac{C_N+\langle\Theta_t^2\rangle D_N}{\langle\Theta_b^2\rangle}\right]^{1/2}\right\} \qquad (3.36)$$

$$C_N(u_1,\ldots,u_{N-1}) = \left(\frac{R-R_c}{R}\right)\left[\sum_{i=1}^{N-1}\langle\Theta_f^2\rangle u_i u_i^2 + \sum_{j=1}^{N-2}\sum_{k=j+1}^{N-1}\frac{\langle\Theta_f^2\rangle u_j\langle\Theta_f^2\rangle u_k}{\langle\Theta_b^2\rangle}(u_j-u_k)^2\right]$$

(3.37a)

$$D_N(u_1,\ldots,u_{N-1}) = 1 + \sum_{i=1}^{N-1}\frac{\langle\Theta_f^2\rangle u_i}{\langle\Theta_b^2\rangle}\left(1-\frac{R-R_c}{R}u_i\right)^2 \qquad (3.37b)$$

$$E_N(u_1,\ldots,u_{N-1}) = 1 + \frac{\langle\Theta_t^2\rangle + \sum_{i=1}^{N-1}\langle\Theta_f^2\rangle u_i}{\langle\Theta_b^2\rangle} \qquad (3.37c)$$

and where

$R-R_c$ — penetration depth to the location of the backscattering event, where R and R_c are the ranges from the lidar to the backscattering event location and the scattering medium boundary, respectively

u — dimensionless penetration depth, $u = x/(R-R_c)$ where $0 \le x \le (R-R_c)$

actually, it should be $\theta_L^2 = \frac{L^2}{R^2}$

See opposite page (also forgot to divide by P_1).

$a(u)$ spatial structure of the fraction of the total scattered energy which is defined by the forward scattering phase function

τ optical penetration depth, $\tau = \int_0^{R-R_c} \beta(u)du$ where β is the optical extinction coefficient

$\rho(u)$ spatial structure of the optical extinction coefficient about its mean value, $\rho(u) = \beta(u)\cdot(R-R_c)/\tau$

ψ receiver half width field of view

L lateral separation between the lidar transmitter and receiver optical axes

$<\Theta_b^2>$ mean square angle for backscattering by the atmosphere at range R

$<\Theta_f^2>_u$ mean square angle for forward scattering by the atmosphere at the dimensionless penetration depth u

$<\Theta_t^2>$ mean square angle of the transmitter beam pattern

$I_o(\ )$ modified Bessel function (cf. Abramowitz and Stegun (1965), Eq. 9.6.16).

When the lidar system is coaxial such that $L = 0$, then Eq. 3.36 reduces to the relatively simple result

$$\mathcal{I}_N = \frac{\prod_{i=1}^{N-1} a(u_i)\rho(u_i)}{D_N} \{1-\exp[\frac{-D_N\psi^2}{C_N+<\Theta_t^2>D_N}]\} \quad (3.38)$$

$\{1 - \exp[-\frac{\psi^2}{<\Theta_t^2>}]\}^{-1}$

Note that the integrands $\mathcal{I}_N$ of Eqs. 3.36 and 3.38 are <u>independent</u> of the ordering of the scattering events.

The single scatter signal contribution for a non-coaxial lidar is work from Eq. B.13

$$P_{0,0} = \frac{2\gamma_t}{\pi\psi^2} \frac{\Pi_{0,s,0}}{\Delta_{0,s,0} B_{0,s,0}} \int_0^{\psi} \theta d\theta \exp\left\{ -\left[\frac{(A_{0,s,0}+R_c)^2}{B_{0,s,0}} + \frac{\Gamma_{0,s,0}}{\Delta_{0,s,0}} \right] \theta^2 - \frac{L^2}{B_{0,s,0}} \right\} \cdot$$

$$\cdot I_0\left(2L\theta \frac{(A_{0,s,0}+R_c)^2}{B_{0,s,0}} \right)$$

where

$$\Pi_{0,s,0} = \beta(R-R_c) \sum_s \frac{P_s(\pi)}{4\pi} \exp\left\{ -2\int_0^{R-R_c} \beta(x)dx \right\}$$

$$\Delta_{0,s,0} = \gamma_{v,s} + \gamma_t$$

$$\Gamma_{0,s,0} = \gamma_{v,s} \cdot \gamma_t$$

$$A_{0,s,0} + R_c = (R-R_c+R_c) - \frac{\gamma_{v,s}}{\gamma_{v,s}+\gamma_t}(R-R_c+R_c) = R\frac{\gamma_t}{\gamma_{v,s}+\gamma_t}$$

$$B_{0,s,0} = B_{0,0,0} + \frac{\Delta_{0,0,0}}{\Delta_{0,s,0}} A^2_{0,0,0} = \frac{R^2}{\gamma_{v,s}+\gamma_t}$$

thus

$$P_{0,0} = \frac{2\beta(R)\frac{P(\pi)}{4\pi}\gamma_t}{\pi\psi^2 R^2} \exp\left(-2\int_0^R \beta dx\right) \cdot \int_0^{\psi} \theta d\theta \exp\left\{ -\gamma_t\theta^2 - \frac{L^2}{R^2}(\gamma_v+\gamma_t) \right\} \cdot$$

$$\cdot I_0\left(2\frac{L}{R}\gamma_t\theta \right)$$

let $L=0$, then

$$P_{0,0} = \frac{\beta(R)\frac{P(\pi)}{4\pi}}{\pi\psi^2R^2} e^{-2\int_0^R \beta dx} \cdot \underbrace{\int_0^{\psi} e^{-\gamma_t\theta^2} 2\gamma_t\theta d\theta}_{\exp\{-\gamma_t\psi^2\}}$$

This is opposite page 84 (Eq. 3.39a)

$$P_1(R) = \frac{2\beta(R)\,\frac{\mathbf{P}(\pi)}{4\pi}}{\pi\psi^2\langle\Theta_t^2\rangle R^2}\exp\left[-2\int_0^R \beta(R')dR'\right]\cdot\int_0^\psi \theta d\theta \,\exp\left[-\frac{\theta^2}{\langle\Theta_t^2\rangle} - \frac{L^2}{R^2}\frac{1}{\langle\Theta_t^2\rangle+\langle\Theta_b^2\rangle}\right]\cdot \mathbf{I}_o\left[2\,\frac{L}{R}\,\frac{\Theta}{\langle\Theta_t^2\rangle}\right] \qquad (3.39a)$$

When the lidar transmitter and receiver optical axes are coaxial, then $L = 0$ and Eq. 3.39a reduces to

$$P_1(R) = \frac{\beta(R)\,\frac{\mathbf{P}(\pi)}{4\pi}}{\pi\psi^2 R^2}\exp\left[-2\int_0^R \beta(R')dR'\right]\cdot\left\{1-\exp\left[-\frac{\psi^2}{\langle\Theta_t^2\rangle}\right]\right\} \qquad (3.39b)$$

The Neumann solution [Eqs. 3.35-3.39] is identical to the ray tracing theory of section 3.1 for the return signal of a coaxial monostatic lidar, particularly to Eq. 3.2 for isotropic backscattering and to Eq. 3.10 for double scattering with no transmitter beam divergence. This Neumann solution is given in a form for scattering phase functions which are characterized by single Gaussian functions. The solution can be generalized to multi-Gaussian phase functions by incorporating the appropriate combinations of $a(u_i)$, $\langle\Theta_f^2\rangle_{u_i}$ and $\langle\Theta_b^2\rangle$ into the formulae for ϑ_n.

The general solution for multiple scattering can be significantly simplified when the components $P_{n,m}$ are related to $P_{n+m,0}$ by Eq. 3.4. A test of Eq. 3.4 was

$\left(\frac{\langle \mathbb{P}(\pi)\rangle_N}{\mathbb{P}(\pi)} = 1\right)$. The coefficients A_N are given as a function of the dimensionless receiver field of view T^2 in Figs. 3.2 and 3.3, and in Table 3.1. The average value of the backscatter phase function $\frac{\langle \mathbb{P}(\pi)\rangle_N}{\mathbb{P}(\pi)}$ is then a complicated function of the remaining solution parameters such as phase function anisotropy and spatial inhomogeneity of the scattering medium. The value of the average backscatter phase function is examined in chapter 4.

The general formula for the contribution of multiple small angle scattering to the return signal of a coaxial monostatic lidar is

$$\frac{P_N}{P_1} = \tau^{N-1}\int_0^1 du_1 \int_0^{u_1} du_2 \ldots \int_0^{u_{N-2}} du_{N-1} \cdot \mathcal{J}_N \tag{3.35}$$

where

$$\mathcal{J}_N = \frac{\prod_{i=1}^{N-1} a(u_i)\rho(u_i)}{D_N} \left\{1-\exp\left[\frac{-D_N\psi^2}{C_N+\langle\Theta_t^2\rangle D_N}\right]\right\} \tag{~~3.36~~ 3.38}$$

$\left[1-\exp\left(\frac{-\psi^2}{\langle\Theta_t^2\rangle}\right)\right]^{-1}$

$$C_N(u_1,\ldots,u_{N-1}) = \left(\frac{R-R_c}{R}\right)^2\left[\sum_{i=1}^{N-1}\langle\Theta_f^2\rangle u_i u_i^2 + \sum_{j=1}^{N-2}\sum_{k=j+1}^{N-1}\frac{\langle\Theta_f^2\rangle u_j \langle\Theta_f^2\rangle u_k}{\langle\Theta_b^2\rangle}(u_j-u_k)^2\right] \tag{3.37a}$$

$$D_N(u_1,\ldots,u_{N-1}) = 1 + \sum_{i=1}^{N-1}\frac{\langle\Theta_f^2\rangle u_i}{\langle\Theta_b^2\rangle}\left(1-\frac{R-R_c}{R}u_i\right)^2 \tag{3.37b}$$

eq. B.6c check (substitution)

$$\hat{e}_0(x_1, \zeta) = \frac{\gamma_t}{2\pi^2} \frac{e^{-\int_0^{x_1} \beta dx}}{1} \exp\left\{ -\frac{\gamma_t}{1}(\eta^2+\zeta^2) + \hat{j}(\eta p + \zeta q)(x_1 + R_c) \right\}$$

This is opposite page 183 (Eq. B.6c)

typo

$$\Box_{n,s,0} = \exp[-\int_{x_{n+1}}^{R-R_c} \beta(x)dx]\cdot\beta(R-R_c)\sum_{s=1}^{S}\frac{P_s(\pi)}{4\pi}\cdot\Box_{n,0,0} \qquad \text{(B.5b)}$$

$$\Box_{\ell,0,0} = \exp[-\int_{x_\ell}^{x_{\ell+1}} \beta(x)dx]\cdot\int_{0}^{x_{\ell+1}} dx_\ell\,\beta(x_\ell)\cdot\sum_{i_\ell=1}^{I} a_{i_\ell}(x_\ell)\gamma_{fi_\ell}(x_\ell)\Box_{\ell-1,0,0}$$

$$\text{for } 1\le\ell\le n \text{ and } x_{n+1}=R-R_c \qquad \text{(B.5c)}$$

$$\Box_{0,0,0} = \exp[-\int_{0}^{x_1} \beta(x)dx] \qquad \text{(B.5d)}$$

The symbols ξ and ξ' are used to represent the parameter sets $[p,q,\eta,\zeta]$ and $[p,q,\eta',\zeta']$, respectively. From Eq. 3.23

$$\hat{i}_o(0,\xi) = \frac{\gamma_t}{2\pi^2}\exp\{-\gamma_t(\eta^2+\zeta^2)+\hat{j}(\eta p+\zeta q)R_c\} \qquad \text{(B.6a)}$$

From Eq. 3.24a

$$\hat{i}_o(x_1,\xi) = \hat{i}_o(0,\xi)\exp\{-\int_{0}^{x_1}\beta dx+\hat{j}(\eta p+\zeta q)x_1\} \qquad \text{(B.6b)}$$

$$\boxed{\hat{i}_o(x_1,\xi) = \frac{\gamma_t}{2\pi^2}\frac{\Box_{0,0,0}}{\Delta_{0,0,0}}\exp\{-\frac{\Gamma_{0,0,0}}{\Delta_{0,0,0}}(\eta^2+\zeta^2)+\hat{j}(\eta p+\zeta q)A_{0,0,0}\}} \qquad \text{(B.6c)}$$

From Eqs. 3.24b and 3.24c

This is opposite page 184

Note a_{i_n} & γ_{f,i_n} ⇒ evaluation at x_n

check B.76 by substitution:

$$\hat{e}_1(x_2,\xi) = \mathbb{A}\cdot\exp\left(-\int_{x_1}^{x_2}\beta(x)dx\right)\cdot\int_0^{x_2} dx_1\,\beta(x_1)\sum_{i_1=1}^{I} a_{i_1}(x_1)\gamma_{f,i_1}(x_1)$$

where

$$\mathbb{A} = \frac{\gamma}{2\pi^2}\cdot\frac{\exp\left(-\int_0^{x_1}\beta(x)dx\right)}{\gamma_{f,i_1}+\gamma_t}\cdot\exp\Big\{-\frac{\gamma_{f,i_1}\gamma_t}{\gamma_{f,i_1}+\gamma_t}(\eta^2+\zeta^2) +$$

$$+\hat{j}(\eta p+\zeta q)\left[(x_2-x_1)+\gamma_{f,i_1}\frac{1}{\gamma_{f,i_1}+\gamma_t}(x_1+R_c)\right]$$

$$-\frac{1}{4}(p^2+q^2)\frac{(x_1+R_c)^2}{\gamma_{f,i_1}+\gamma_t}\Big\}$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{e}_0(x_1,\varepsilon')\cdot\exp\{-\gamma_{f,i_1}[(\eta'-\eta)^2+(\zeta'-\zeta)^2]+\hat{j}(\eta p+\zeta q)(x_2-x_1)]\}d\eta'd\zeta'$$

i.e.

$$\frac{1}{\pi}\exp[-\gamma_{f,i_1}(\eta^2+\zeta^2)]\iint_{-\infty}^{+\infty}d\eta'd\zeta'\exp\Big\{-(\gamma_t+\gamma_{f,i_1})(\eta'^2+\zeta'^2)$$

$$+\eta'(\hat{j}p(x_1+R_c)+2\gamma_{f,i_1}\eta)+\zeta'(\hat{j}q(x_1+R_c)+2\gamma_{f,i_1}\zeta)\Big\}$$

$$=\frac{1}{\gamma_{f,i_1}+\gamma_t}\exp\Big\{-\frac{\gamma_{f,i_1}\gamma_t}{\gamma_{f,i_1}+\gamma_t}(\eta^2+\zeta^2)+\hat{j}(\eta p+\zeta q)\frac{\gamma_{f,i_1}}{\gamma_{f,i_1}+\gamma_t}(x_1+R_c)$$

$$-\frac{1}{4}(p^2+q^2)\frac{(x_1+R_c)^2}{\gamma_{f,i_1}+\gamma_t}\Big\}$$

see identity on p 77

$$\frac{e^{-\gamma_f\eta^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty}d\eta'\exp\{-(\gamma_t+\gamma_f)\eta'^2+\eta'(\hat{j}p(x_1+R_c)+2\gamma_f\eta)\}$$

$$=\frac{e^{-\gamma_f\eta^2}}{\sqrt{\gamma_f+\gamma_t}}\exp\left\{\frac{-p^2(x_1+R_c^2)+4\hat{j}\gamma_f\eta p(x_1+R_c)+4\gamma_f^2\eta^2}{4(\gamma_f+\gamma_t)}\right\}$$

$$=\frac{1}{(\gamma_f+\gamma_t)^{1/2}}\exp\left\{-\eta^2\cdot\frac{-\gamma_f^2+\gamma_f(\gamma_f+\gamma_t)}{\gamma_f+\gamma_t}+\hat{j}\frac{\gamma_f}{\gamma_f+\gamma_t}\eta p(x_1+R_c)-\frac{1}{4}p^2\frac{(x_1+R_c)^2}{\gamma_f+\gamma_t}\right\}.$$

O.K.

$$\hat{i}_1(x_2,\boldsymbol{\zeta}) = \int_0^{x_2} dx_1\ \beta(x_1) \sum_{i_1=1}^{I} \frac{a_{i_1}\gamma_{f,i_1}}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{i}_o(x_1,\boldsymbol{\zeta}')\cdot\exp\{-\gamma_{f,i_1}[(\eta'-\eta)^2+(\zeta'-\zeta)^2]$$

$$- \int_{x_1}^{x_2} \beta dx + \hat{j}(\eta p+\zeta q)(x_2-x_1)\}d\eta' d\zeta' \qquad \text{(B.7a)}$$

or

$$\boxed{\hat{i}_1(x_2,\boldsymbol{\zeta}) = \frac{\gamma_t}{2\pi^2}\frac{\Box_{1,0,0}}{\Delta_{1,0,0}} \exp\{-\frac{\Gamma_{1,0,0}}{\Delta_{1,0,0}}(\eta^2+\zeta^2)+\hat{j}(\eta p+\zeta q)A_{1,0,0} - \frac{1}{4}(p^2+q^2)B_{1,0,0}\}} \qquad \text{(B.7b)}$$

By induction, it follows that

$$\boxed{\hat{i}_n(R-R_c,\boldsymbol{\zeta}) = \frac{\gamma_t}{2\pi^2}\frac{\Box_{n,0,0}}{\Delta_{n,0,0}} \exp\{-\frac{\Gamma_{n,0,0}}{\Delta_{n,0,0}}(\eta^2+\zeta^2) + \hat{j}(\eta p+\zeta q)A_{n,0,0} - \frac{1}{4}(p^2+q^2)B_{n,0,0}\}} \qquad \text{(B.8)}$$

Now perform the large angle scattering at $R-R_c$. From Eq. 3.27

$$\hat{\hat{i}}_{n,0}(x_{n+1},\boldsymbol{\zeta}) = \beta(R-R_c) \sum_{s=1}^{S} \frac{P_s(\pi)}{4\pi} \cdot\exp\{-\int_{x_{n+1}}^{R-R_c} \beta dx + \hat{j}(\eta p+\zeta q)(x_o-x_{n+1})\}\cdot$$

$$\cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{i}_n(R-R_c,\boldsymbol{\zeta}')\ \exp\{-\gamma_{b,s}[(\eta'+\eta)^2+(\zeta'+\zeta)^2]\}d\eta' d\zeta' \qquad \text{(B.9a)}$$

This is opposite page 186

$\eta = \theta\cos\phi$

$\zeta = \theta\sin\phi$

$\eta^2+\zeta^2 = \theta^2(\cos^2\phi+\sin^2\phi) = \theta^2$

$\eta+\zeta = \theta\,(\cos\phi+\sin\phi)$

$\cos\phi+\sin\phi = \cos\phi + \cos(\pi/2-\phi)$

$= 2\cos\left(\frac{\phi+\pi/2-\phi}{2}\right)\cos\left(\frac{\phi-\pi/2+\phi}{2}\right)$ A&S 4.3.36

$= 2\cos(\pi/4)\cos(\phi-\pi/4)$

$= \sqrt{2}\cos(\phi-\pi/4)$

we must integrate over ϕ:

$\int_0^{2\pi} d\phi \exp[-a\theta(\cos\phi+\sin\phi)]$, where $\boxed{a = \sqrt{2}L \cdot \frac{A_{n,s,m}+R_c}{B_{n,s,m}}}$

$= \int_0^{2\pi} d\phi \exp[-\sqrt{2}\,a\theta\cos(\phi-\pi/4)]$

$= 2\int_0^{\pi} d\phi \exp[\pm\sqrt{2}\,a\theta\cos\phi]$

$= 2\pi\, I_0(\sqrt{2}\,a\theta) = 2\pi\, I_0\left(2L\theta\frac{A_{n,s,m}+R_c}{B_{n,s,m}}\right)$

$$P_{n,m} = \int_{-\psi}^{\psi}\int_{-\psi}^{\psi} d\eta d\zeta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dpdq \; \hat{\hat{i}}_{n,m}(x=0,\boldsymbol{\xi}) \cdot \tilde{r}(x=0,\boldsymbol{\xi}) \tag{B.12a}$$

or

$$P_{n,m} = \int_{-\psi}^{\psi}\int_{-\psi}^{\psi} d\eta d\zeta \; \frac{\gamma_t}{4\pi^3\psi^2} \frac{\square_{n,s,m}}{\Delta_{n,s,m}} \exp\{-\frac{\Gamma_{n,s,m}}{\Delta_{n,s,m}}(\eta^2+\zeta^2)\} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dpdq \cdot$$

$$\cdot \exp\{-\frac{B_{n,s,m}}{4}(p^2+q^2) + \hat{j}p[\eta(A_{n,s,m}+R_c)+\frac{L}{\sqrt{2}}] + \hat{j}q[\zeta(A_{n,s,m}+R_c)+\frac{L}{\sqrt{2}}]\} \tag{B.12b}$$

Integrating over (p,q) and transforming the normal coordinates (η,ζ) to spherical coordinates (θ,φ), the final solution is

$$P_{n,m} = \frac{2\gamma_t}{\pi\psi^2} \frac{\square_{n,s,m}}{\Delta_{n,s,m} B_{n,s,m}} \int_0^{\psi} \theta d\theta \; \exp\{-[\frac{(A_{n,s,m}+R_c)^2}{B_{n,s,m}} + \frac{\Gamma_{n,s,m}}{\Delta_{n,s,m}}]\theta^2 - \frac{L^2}{B_{n,s,m}}\} \cdot$$

$$\cdot \mathbf{I}_0[2L\theta \frac{(A_{n,s,m}+R_c)}{B_{n,s,m}}] \quad \text{for } L \geq 0 \tag{B.13}$$

where $\mathbf{I}_0(f(\theta))$ is the modified Bessel function [cf. Abramowitz and Stegun (1965), Eq. 9.6.16]. When the lidar transmitter and receiver are coaxial, then $L = 0$ and $\mathbf{I}_0(0) = 1$, and Eq. B.13 is readily integrated over θ with the result

This is opposite page 187

integrating Eq. B.12b over $dp\,dq$ only, then

$$P_{m,m} = \frac{\gamma_t}{\pi^2 \psi^2} \iint\limits_{(\eta-\eta_0)^2+(\zeta-\zeta_0)^2 \le \psi^2} d\eta\, d\zeta \; \frac{\square_N}{\Delta_N B_N} \exp\left\{ -(\eta^2+\zeta^2)\left[\frac{I_N}{\Delta_N} + \frac{(A_N+R_c)^2}{B_N}\right] - \frac{L^2}{B_N} + \sqrt{2}\, L \frac{(A_N+R_c)}{B_N}(\eta+\zeta) \right\}$$

this simplifies to Eq. B.13 when $\eta_0 = \zeta_0 = 0$.

$$P_{n,m} = \frac{\gamma_t}{\pi\psi^2} \Box_{n,s,m} \frac{1 - \exp\{-[\frac{(A_{n,s,m}+R_c)^2}{B_{n,s,m}} + \frac{\Gamma_{n,s,m}}{\Delta_{n,s,m}}]\psi^2\}}{\Delta_{n,s,m}(A_{n,s,m}+R_c)^2 + \Gamma_{n,s,m}B_{n,s,m}} \tag{B.14}$$

for $L = 0$ and where

$$\Box_{n,s,m} = \exp[-2\int_0^{R-R_c} \beta(x)dx]\,\beta(R-R_c) \sum_{s=1}^{S} \frac{P_s(\pi)}{4\pi} \int_0^{R-R_c} dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_2 \,\cdot$$

$$\cdot \int_0^{R-R_c} dx_{n+1} \int_0^{x_{n+1}} dx_{n+2} \cdots \int_0^{x_{n+m-1}} dx_{n+m} \prod_{k=1}^{n+m} \beta(x_k) \cdot \sum_{i_k=1}^{I} a_{i_k}(x_k)\gamma_{f,i_k}(x_k) \tag{B.15}$$